AI Applications Lecture 2

Machine Learning Models and Parametric Functions

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Outline

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Introduction

1.1 Review of the Previous Lecture

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- The tasks we want computers to solve can be formulated as an **input-output relationship**.
- Real-world entities such as text, images, and audio can be converted into numerical data (vectors or tensors).
- Thus, solving a task with AI boils down to a **function determination problem**: finding an appropriate **function** from an input space to an output space.

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- Formulate a function determination problem as a parameter determination problem for a **parametric function**.
- Explain what parameters and functions constitute basic AI/machine learning models such as linear regression, logistic regression, decision trees, and k-NN.
- Formulate machine learning as determining parameters using data, and explain the difference between **training** and **inference**.

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Therefore, this lecture emphasizes understanding more general and universal concepts that will be applicable in the future and will not become obsolete, rather than learning individual technologies.

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Understanding this concept will allow you to:

- View various models from a unified perspective.
- Clearly distinguish between the two major phases of AI development: **training** and **inference**.

This distinction is an essential foundation for understanding neural network applications and practical issues such as licensing.

Preparation: Mathematical Notations

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Here is a list of the basic mathematical notations used in this lecture.

- **Definition:**
  - $(\mathrm{LHS}) \coloneqq (\mathrm{RHS})$: The left-hand side is defined by the right-hand side.

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- **Definition:**
  - $(\mathrm{LHS}) \coloneqq (\mathrm{RHS})$: The left-hand side is defined by the right-hand side.
- **Set:** Denoted by uppercase calligraphic letters (e.g., $\mathcal{A}$).
  - $x \in \mathcal{A}$: element x belongs to set $\mathcal{A}$.
  - $\{a, b, c\}$: The set consisting of elements a, b, c.
  - $\{x \in \mathcal{A} | P(x)\}$: The set of elements in $\mathcal{A}$ for which $P(x)$ is true.
  - $\mathbb{R}$ (reals), $\mathbb{Z}$ (integers), with subscripts like $_{>0}$ (positive) or $_{\geq 0}$ (non-negative).

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- **Function:**
  - $f : X \to Y$: f is a map from set X to set Y.
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- **Vector:** Denoted by bold italic lowercase letters (e.g., $\boldsymbol{v}$).
  - A vector is a column of numbers.
  - $\boldsymbol{v} \in \mathbb{R}^n$: an n-dimensional real vector.
  - The i-th element is v_i. $\boldsymbol{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^\top$.

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- **Matrix:** Denoted by bold italic uppercase letters (e.g., $\boldsymbol{A}$).
  - $\boldsymbol{A} \in \mathbb{R}^{m,n}$: an $m \times n$ real matrix.
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- **Tensor:**
  - In this lecture, a tensor is simply a multi-dimensional array.
  - Vector = 1st-order tensor, Matrix = 2nd-order tensor.
  - Higher-order tensors are denoted by $\underline{\boldsymbol{A}}$.

Parametric Functions

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Let's formalize this idea.

Definition (Parametric Function)

If there exists a set Θ (the **parameter space**), and for each element $\boldsymbol{\theta} \in \Theta$, a function $f_{\boldsymbol{\theta}} : \mathcal{X} \to \mathcal{Y}$ is defined, then the family of functions $(f_{\boldsymbol{\theta}})_{\boldsymbol{\theta} \in \Theta}$ is called a **parametric function**. Each $\boldsymbol{\theta}$ is called a **parameter**.

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Why look at models other than neural networks?

- **Linear/Logistic Regression**: Simple, foundational, and a special case of neural networks.
- **Decision Trees**: A different kind of function not representable by typical NNs. Parameters are not a fixed-size vector. Intuitive and powerful.
- **k-Nearest Neighbors (k-NN)**: Another non-NN example. Also has variable-size parameters. Extremely simple training algorithm, which is great for illustrating the concepts of training and inference.

3.1 Example 1: Linear Regression

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Example

For $d_{\text{in}} = 2$, parameters $\boldsymbol{w} = \begin{bmatrix} 1.0 \\ -2.0 \end{bmatrix}, b = 0.5$, and input $\boldsymbol{x} = \begin{bmatrix} 3.0 \\ 4.0 \end{bmatrix}$, the output is:

$$\begin{aligned} y = f_{(\boldsymbol{w},b)}(\boldsymbol{x}) &= (1.0 \times 3.0) + (-2.0 \times 4.0) + 0.5 \\ &= 3.0 - 8.0 + 0.5 = -4.5 \end{aligned}$$

Exercise

For parameters $\boldsymbol{w} = \begin{bmatrix} -0.5 \\ 3.0 \end{bmatrix}, b = -1.0$, calculate the output for the same input as in the previous example, $\boldsymbol{x} = \begin{bmatrix} 3.0 \\ 4.0 \end{bmatrix}$.

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Answer:

$$y = f_{(\boldsymbol{w},b)}(\boldsymbol{x}) = (-0.5 \times 3.0) + (3.0 \times 4.0) + (-1.0)$$
$$= -1.5 + 12.0 - 1.0 = 9.5$$

(This shows that the output changes with different parameters for the same input.)

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- **Parameters:** Same as linear regression, $\boldsymbol{\theta} \coloneqq (\boldsymbol{w}, b)$.
- **Function:** $f_{\boldsymbol{w},b}(\boldsymbol{x}) \coloneqq \sigma(\boldsymbol{w}^\top \boldsymbol{x} + b)$.
- Here, $\sigma(z) \coloneqq \frac{1}{1+e^{-z}}$ is the **sigmoid function**.

3.2 Example 2: Logistic Regression

Example

Using the same parameters and input as the first linear regression example ($\boldsymbol{w} = \begin{bmatrix} 1.0 \\ -2.0 \end{bmatrix}, b = 0.5, \boldsymbol{x} = \begin{bmatrix} 3.0 \\ 4.0 \end{bmatrix}$), we first calculate the linear part $z = \boldsymbol{w}^\top \boldsymbol{x} + b = -4.5$.

The final output is:

$$\begin{aligned} y = \sigma(-4.5) &= \frac{1}{1 + e^{4.5}} \\ &\approx \frac{1}{1 + 90.017} \approx 0.011 \end{aligned}$$

3.2 Example 2: Logistic Regression

Exercise

For the same input $\boldsymbol{x} = \begin{bmatrix} 3.0 \\ 4.0 \end{bmatrix}$, calculate the output using the parameters from the linear regression exercise, $\boldsymbol{w} = \begin{bmatrix} -0.5 \\ 3.0 \end{bmatrix}, b = -1.0$.

Exercise

For the same input $\boldsymbol{x} = \begin{bmatrix} 3.0 \\ 4.0 \end{bmatrix}$, calculate the output using the parameters from the linear regression exercise, $\boldsymbol{w} = \begin{bmatrix} -0.5 \\ 3.0 \end{bmatrix}, b = -1.0$.

Answer: First, the linear part is $z = \boldsymbol{w}^\top \boldsymbol{x} + b = 9.5$. The final output is:

$$\begin{aligned} y = \sigma(9.5) &= \frac{1}{1 + e^{-9.5}} \\ &\approx \frac{1}{1 + 0.0000748} \approx 0.999925 \end{aligned}$$

3.3 Example 3: Decision Tree / Regression Tree

Decision trees are examples where parameters are not a fixed-length vector. For an input $\boldsymbol{x} \in \mathbb{R}^{d_{\text{in}}}$, the output is determined by traversing a tree structure.

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Mathematical Formulation:

- A binary tree structure with nodes indexed by integers.
- **Parameters θ define:**
  - The set of nodes in the tree ($\mathcal{V}$).
  - For each **internal node**: which input feature to check (j_i) and a threshold (t_i).
  - For each **leaf node**: the output value (c_l).
- **Function $f_{\boldsymbol{\theta}}(\boldsymbol{x})$:**
  - Start at the root node.
  - At each internal node, compare x_{j_i} with t_i to decide whether to go left or right.
  - When a leaf node is reached, return its value c_l.

3.3 Example 3: Regression Tree

Example (Regression Tree)

Let $d_{\text{in}} = 2$. Consider a tree defined by parameters $\boldsymbol{\theta}_1$:

- Internal nodes:
  - Node 1: split on $x_1 < 2.5$
  - Node 3: split on $x_2 < 0.0$
- Leaf nodes:
  - Node 2: value = 10.0
  - Node 6: value = -5.0
  - Node 7: value = 8.0

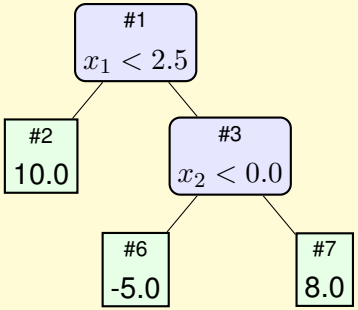


Figure 1: Parameters $\boldsymbol{\theta}_1$

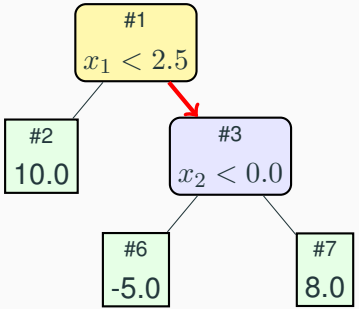
Calculate the output for input $\boldsymbol{x} = \begin{bmatrix} 3.0 \\ -1.0 \end{bmatrix}$.

3.3 Example 3: Regression Tree

Input: $\boldsymbol{x} = \begin{bmatrix} 3.0 \\ -1.0 \end{bmatrix}$

Step 1: Evaluate at root node 1.

- The rule is: $x_1 < 2.5$.
- Our input has $x_1 = 3.0$.
- Since $3.0 \geq 2.5$, the condition is false.
- We proceed to the **right** child, node 3.

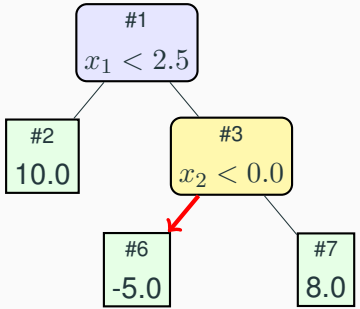


3.3 Example 3: Regression Tree

Input: $x = \begin{bmatrix} 3.0 \\ -1.0 \end{bmatrix}$

Step 2: Evaluate at node 3.

- The rule is: $x_2 < 0.0$.
- Our input has $x_2 = -1.0$.
- Since $-1.0 < 0.0$, the condition is true.
- We proceed to the **left** child, node 6.



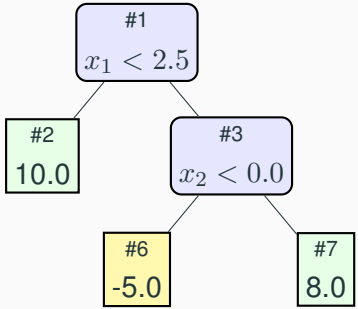
3.3 Example 3: Regression Tree

Input: $\boldsymbol{x} = \begin{bmatrix} 3.0 \\ -1.0 \end{bmatrix}$

Step 3: Arrive at leaf node 6.

- Node 6 is a leaf node.
- The value associated with this leaf is $c_6 = -5.0$.
- This is our final output.

Therefore, the output is $y = f_{\boldsymbol{\theta}_1}(\boldsymbol{x}) = -5.0$.



Exercise

Consider a different tree defined by parameters $\boldsymbol{\theta}_2$. Calculate the output for the same input $\boldsymbol{x} = \begin{bmatrix} 3.0 \\ -1.0 \end{bmatrix}$.

- Internal nodes:
  - Node 1: $x_2 < 5.0$
  - Node 2: $x_1 < 1.0$
- Leaf nodes:
  - Node 3: 20.0
  - Node 4: 1.5
  - Node 5: 3.0

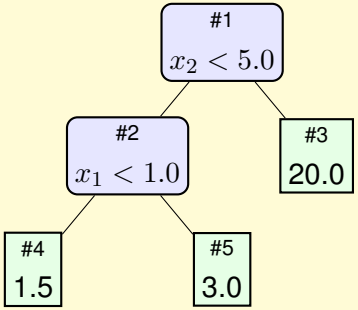


Figure 2: Parameters $\boldsymbol{\theta}_2$

3.3 Example 3: Regression Tree

Answer:

1. At node 1 (root): $x_2 = -1.0 < 5.0$ (true) $\rightarrow$ go left to node 2.
2. At node 2: $x_1 = 3.0 \geq 1.0$ (false) $\rightarrow$ go right to node 5.
3. Node 5 is a leaf node. Its value is $c_5 = 3.0$.

Therefore, the output is $y = f_{\boldsymbol{\theta}_2}(\boldsymbol{x}) = 3.0$.

3.4 Example 4: k-Nearest Neighbors (k-NN)

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- **Parameters:** The training dataset itself, $\boldsymbol{\theta} \coloneqq \mathcal{D} = ((\boldsymbol{x}_1, y_1), \ldots, (\boldsymbol{x}_N, y_N))$, and the number of neighbors, k.

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- **Parameters:** The training dataset itself, $\boldsymbol{\theta} \coloneqq \mathcal{D} = ((\boldsymbol{x}_1, y_1), \ldots, (\boldsymbol{x}_N, y_N))$, and the number of neighbors, k.
- **Function:** For a new input $\boldsymbol{x}_{new}$:
  1. Calculate the distance from $\boldsymbol{x}_{new}$ to every $\boldsymbol{x}_i$ in the dataset.
  2. Find the k data points $(\boldsymbol{x}_i, y_i)$ with the smallest distances.
  3. Aggregate their outputs y_i to get the final result. (e.g., mean for regression, majority vote for classification).

Example (Regression)

Let $k = 3$. The parameters $\boldsymbol{\theta}_1$ are the dataset $\mathcal{D}_1$:

$$\mathcal{D}_1 = \left(\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 5 \right), \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, 10 \right), \left(\begin{bmatrix} 3 \\ 0 \end{bmatrix}, 20 \right), \left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}, 25 \right) \right)$$

Calculate the output for a new input $\boldsymbol{x}_{new} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

3.4 Example 4: k-NN Regression

Input: $\boldsymbol{x}_{new} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. **Data:** $\mathcal{D}_1$.

Step 1: Calculate squared Euclidean distance to each data point.

- $d(\boldsymbol{x}_{new}, \boldsymbol{x}_1)^2 = (2-0)^2 + (1-0)^2 = 4 + 1 = 5$
- $d(\boldsymbol{x}_{new}, \boldsymbol{x}_2)^2 = (2-1)^2 + (1-2)^2 = 1 + 1 = 2$
- $d(\boldsymbol{x}_{new}, \boldsymbol{x}_3)^2 = (2-3)^2 + (1-0)^2 = 1 + 1 = 2$
- $d(\boldsymbol{x}_{new}, \boldsymbol{x}_4)^2 = (2-4)^2 + (1-3)^2 = 4 + 4 = 8$

Step 2: Find the top $k = 3$ nearest neighbors.

- The distances are: 5, 2, 2, 8.
- The smallest three distances correspond to data points $\boldsymbol{x}_2$ (dist 2), $\boldsymbol{x}_3$ (dist 2), and $\boldsymbol{x}_1$ (dist 5).
- Their corresponding outputs are $y_2 = 10, y_3 = 20, y_1 = 5$.

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- Their corresponding outputs are $y_2 = 10, y_3 = 20, y_1 = 5$.

Step 3: Aggregate the outputs (mean for regression).

- We calculate the average of the neighbors' output values.

$$y = \frac{10 + 20 + 5}{3} = \frac{35}{3} \approx 11.67$$

The final output is $y \approx 11.67$.

Exercise

Let $k = 2$ and assume the parameters $\boldsymbol{\theta}_2$ consist of the following five data points.

$$\mathcal{D}_2 = ((\begin{bmatrix}1\\1\end{bmatrix}, 1), (\begin{bmatrix}2\\3\end{bmatrix}, 4), (\begin{bmatrix}-1\\2\end{bmatrix}, 6), (\begin{bmatrix}0\\-1\end{bmatrix}, 8), (\begin{bmatrix}4\\0\end{bmatrix}, 12))$$

Calculate the output for the same new input as in the previous example, $\boldsymbol{x}_{new} = \begin{bmatrix}2\\1\end{bmatrix}$.

Answer:

- Squared distances are: 1, 4, 10, 8, 5.
- The two nearest neighbors are $(\begin{bmatrix}1\\1\end{bmatrix}, 1)$ and $(\begin{bmatrix}2\\3\end{bmatrix}, 4)$.
- Their outputs are $\{1, 4\}$.
- The mean is $y = \frac{1+4}{2} = 2.5$.

Remark

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In this lecture, we consider non-parametric models as a type of "parametric function with a variable-length data structure as parameters."

Scheme of Training and Inference

4. Scheme of Training and Inference

Viewing models as parametric functions, $(f_{\boldsymbol{\theta}})_{\boldsymbol{\theta}\in\Theta}$, separates the problem-solving process into two distinct phases.

- **Training:** The process of finding the "optimal" parameters $\boldsymbol{\theta}^* \in \Theta$ that can best solve the task, using given **training data** $\mathcal{D}$. This is often called **fitting**.

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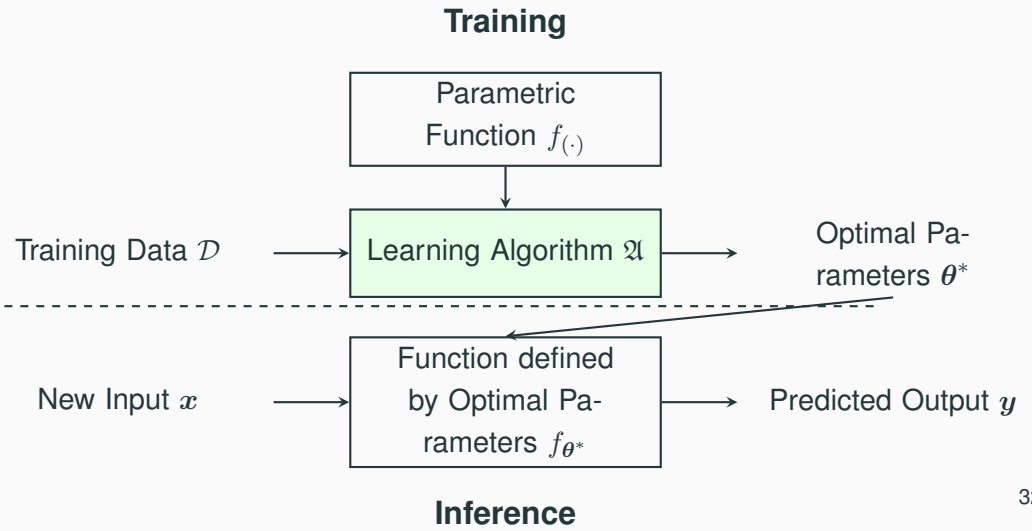
- **Inference:** The process of fixing the parameters $\boldsymbol{\theta}^*$ and using the specific function $f_{\boldsymbol{\theta}^*}$ to calculate the output $\boldsymbol{y}$ for a new, unknown input $\boldsymbol{x}$. This is also called **prediction**.

$$\boldsymbol{y} = f_{\boldsymbol{\theta}^*}(\boldsymbol{x})$$

4. Scheme of Training and Inference



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- The license for the inference part and the training part may not be the same.
- Handling neural network models without understanding their licenses carries legal risks.

Specific Examples of Learning Algorithms

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Inference is simply the computation $\boldsymbol{y} = f_{\boldsymbol{\theta}^*}(\boldsymbol{x})$. All the calculation examples we have seen so far correspond to this inference phase.

The training phase, finding the "optimal" parameters $\boldsymbol{\theta}^*$ from data $\mathcal{D}$, differs greatly depending on the type of parametric function.

The goal here is not to memorize algorithms, but to understand that training is the process of determining parameters, and it's different from inference.

5.1 Example 1: Training of k-NN

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Training is completed just by "memorizing the data."

Models like this, where most of the computation is deferred until inference time, are sometimes called **lazy learning** models.

5.2 Example 2: Training of Linear Regression

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Training is often formulated as an **optimization problem**: "minimizing a function that measures the 'badness' of the model's predictions on the training data."

Definition (Loss Function)

A function $L(\boldsymbol{\theta})$ that takes parameters $\boldsymbol{\theta}$ and outputs a non-negative real value representing how "bad" the model is for the data is called a **cost function**, **loss function**, or **objective function**. The goal is to find:

$$\boldsymbol{\theta}^* \coloneqq \arg\min_{\boldsymbol{\theta} \in \Theta} L(\boldsymbol{\theta})$$

5.2.1 Least Squares Method

For linear regression, a widely used loss function is the **Sum of Squared Residuals (SSR)**.

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Definition (Sum of Squared Residuals)

Given training data $\mathcal{D} = ((\boldsymbol{x}_i, y_i))_{i=1}^N$, the SSR is the sum of the squares of the differences (the **residuals**) between the actual outputs y_i and the model's predicted values $f(\boldsymbol{x}_i)$.

$$\begin{aligned} L_{\mathrm{SSR}}(\boldsymbol{w}, b) &:= \sum_{i=1}^{N} (y_i - f_{(\boldsymbol{w},b)}(\boldsymbol{x}_i))^2 \\ &= \sum_{i=1}^{N} (y_i - (\boldsymbol{w}^\top \boldsymbol{x}_i + b))^2 \end{aligned}$$

5.2.2 Least Squares Solution

The least squares solution can be found analytically using linear algebra.

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The least squares solution can be found analytically using linear algebra.

Step 1: Augment vectors and matrices.

- Append a 1 to each input vector: $\tilde{\boldsymbol{x}}_i \coloneqq \begin{bmatrix} \boldsymbol{x}_i \\ 1 \end{bmatrix}$.
- Combine weights and bias: $\tilde{\boldsymbol{w}} \coloneqq \begin{bmatrix} \boldsymbol{w} \\ b \end{bmatrix}$.
- Now, the prediction is a simple dot product: $f(\boldsymbol{x}_i) = \tilde{\boldsymbol{w}}^\top \tilde{\boldsymbol{x}}_i$.

Step 2: Represent data in matrix form.

- Design Matrix $\boldsymbol{X}$: Each row is an augmented input vector $\tilde{\boldsymbol{x}}_i^\top$.
- Target Vector $\boldsymbol{y}$: A column vector of all true outputs y_i.

$$\boldsymbol{X} \coloneqq \begin{bmatrix} \tilde{\boldsymbol{x}}_1^\top \\ \vdots \\ \tilde{\boldsymbol{x}}_N^\top \end{bmatrix}, \quad \boldsymbol{y} \coloneqq \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}$$

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With the matrix formulation, the vector of all predictions $\hat{\boldsymbol{y}}$ is simply $\hat{\boldsymbol{y}} = \boldsymbol{X}\tilde{\boldsymbol{w}}$.

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The loss function SSR becomes the squared L2-norm of the difference vector:

$$L_{\mathrm{SSR}}(\tilde{\boldsymbol{w}}) = \|\boldsymbol{y} - \hat{\boldsymbol{y}}\|_2^2 = \|\boldsymbol{y} - \boldsymbol{X}\tilde{\boldsymbol{w}}\|_2^2$$

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The minimizer of this loss function is the solution to the following equation:

Definition (Normal Equation)

The following equation for $\tilde{\boldsymbol{w}}$ is called the **normal equation**.

$$(\boldsymbol{X}^\top \boldsymbol{X})\tilde{\boldsymbol{w}} = \boldsymbol{X}^\top \boldsymbol{y}. \tag{1}$$

5.2.2 Least Squares Solution

Theorem (Least Squares Solution)

A vector $\tilde{\boldsymbol{w}}$ minimizes the Sum of Squared Residuals (SSR) if and only if it is a solution to the normal equation.

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In particular, if the matrix $\boldsymbol{X}^\top \boldsymbol{X}$ is invertible, the normal equation has a unique solution given by:

$$\tilde{\boldsymbol{w}}^* = (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{y}. \tag{2}$$

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Geometric Interpretation: The least squares method finds the **orthogonal projection** of the target vector $\boldsymbol{y}$ onto the subspace spanned by the columns of the design matrix $\boldsymbol{X}$. The normal equation expresses the condition that the error vector $(\boldsymbol{y} - \boldsymbol{X}\tilde{\boldsymbol{w}})$ must be orthogonal to that subspace.

Example

Let's find the optimal parameters for the following training data $\mathcal{D}_A$:

$$\mathcal{D}_A = ((\boldsymbol{x}_1, y_1) = (\begin{bmatrix} 3 \\ 4 \end{bmatrix}, -4.5), (\boldsymbol{x}_2, y_2) = (\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 1.5), (\boldsymbol{x}_3, y_3) = (\begin{bmatrix} 0 \\ 1 \end{bmatrix}, -1.5))$$

5.2.2 Least Squares Solution

Step 1: Construct the design matrix X and target vector y.

- The input vectors are $\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- The augmented input vectors (add a 1) are $\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.
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- The outputs are $-4.5, 1.5, -1.5$.

$$\boldsymbol{X} = \begin{bmatrix}3 & 4 & 1\\1 & 0 & 1\\0 & 1 & 1\end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix}-4.5\\1.5\\-1.5\end{bmatrix}$$

Step 2: Calculate $\boldsymbol{X}^\top \boldsymbol{X}$ and $\boldsymbol{X}^\top \boldsymbol{y}$.

$$\boldsymbol{X}^\top \boldsymbol{X} = \begin{bmatrix} 3 & 1 & 0 \\ 4 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 10 & 12 & 4 \\ 12 & 17 & 5 \\ 4 & 5 & 3 \end{bmatrix}$$

Step 2: Calculate $\boldsymbol{X}^\top \boldsymbol{X}$ and $\boldsymbol{X}^\top \boldsymbol{y}$. (Continued)

$$\begin{aligned}\boldsymbol{X}^\top \boldsymbol{y} &= \begin{bmatrix} 3 & 1 & 0 \\ 4 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -4.5 \\ 1.5 \\ -1.5 \end{bmatrix} \\ &= \begin{bmatrix} -13.5 + 1.5 \\ -18.0 - 1.5 \\ -4.5 + 1.5 - 1.5 \end{bmatrix} = \begin{bmatrix} -12.0 \\ -19.5 \\ -4.5 \end{bmatrix}\end{aligned}$$

5.2.2 Least Squares Solution

Step 3: Solve the normal equation $(\boldsymbol{X}^\top \boldsymbol{X})\tilde{\boldsymbol{w}} = \boldsymbol{X}^\top \boldsymbol{y}$.

$$\begin{bmatrix} 10 & 12 & 4 \\ 12 & 17 & 5 \\ 4 & 5 & 3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ b \end{bmatrix} = \begin{bmatrix} -12.0 \\ -19.5 \\ -4.5 \end{bmatrix}$$

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Solving this system of linear equations (e.g., by finding the inverse of the matrix on the left), we get the solution:

$$\tilde{\boldsymbol{w}}^* = \begin{bmatrix} w_1 \\ w_2 \\ b \end{bmatrix} = \begin{bmatrix} 1.0 \\ -2.0 \\ 0.5 \end{bmatrix}$$

These are exactly the parameters we used in the first inference example!

5.3 Diversity of Learning Algorithms and Neural Networks

As we have seen, learning algorithms can be completely different:

- **k-NN:** Just store the data ("lazy learning").
- **Linear Regression:** Solve a system of linear equations (the normal equation).
- **Decision Trees:** Use a greedy algorithm to find the best splits (not detailed here).

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In contrast, one of the greatest strengths of **neural networks** is the **uniformity of their learning algorithm**.

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- **Decision Trees:** Use a greedy algorithm to find the best splits (not detailed here).

In contrast, one of the greatest strengths of **neural networks** is the **uniformity of their learning algorithm**.

Most neural networks can be trained under a common framework:

1. Calculate the gradient of the loss function using **backpropagation**.
2. Minimize the loss using **stochastic gradient descent (SGD)** or its variants.

Summary and Future Outlook

6.1 Today's Summary

- A task can be reduced to a parameter determination problem for a **parametric function**.

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- A task can be reduced to a parameter determination problem for a **parametric function**.
- Various AI models like linear regression, decision trees, and k-NN can be understood uniformly as parametric functions.
- Machine learning is separated into a **training** phase (finding optimal parameters from data) and an **inference** phase (using those parameters to make predictions).

6.2 Next Time

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Next time, we will strictly define what a **neural network** is—a particularly expressive type of parametric function. We will use the mathematical tool of a **Directed Acyclic Graph (DAG)** to do this.

Then, we will learn about the distinction between "architecture" and "checkpoints," concepts that are extremely important in the practical application of modern AI.

Appendix: Differentiation by a Vector i

The gradient of a scalar function $f(\boldsymbol{x})$ with respect to a vector $\boldsymbol{x}$ is defined as a vector of partial derivatives.

$$\nabla_{\boldsymbol{x}} f(\boldsymbol{x}) = \frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}} := \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}^\top$$

1. **Derivative of a Linear Term:**

$$\frac{\partial(\boldsymbol{a}^\top \boldsymbol{x})}{\partial \boldsymbol{x}} = \boldsymbol{a}$$

2. **Derivative of a Quadratic Form:** For a symmetric matrix $\boldsymbol{A}$:

$$\frac{\partial(\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x})}{\partial \boldsymbol{x}} = 2\boldsymbol{A}\boldsymbol{x} \quad (\text{if } \boldsymbol{A} = \boldsymbol{A}^\top)$$

Appendix: Differentiation by a Vector ii

Recall the loss function:

$$L(\tilde{\boldsymbol{w}}) = \boldsymbol{y}^\top \boldsymbol{y} - 2\tilde{\boldsymbol{w}}^\top \boldsymbol{X}^\top \boldsymbol{y} + \tilde{\boldsymbol{w}}^\top (\boldsymbol{X}^\top \boldsymbol{X})\tilde{\boldsymbol{w}}$$

Differentiating each term w.r.t. $\tilde{\boldsymbol{w}}$:

- 1st term ($\boldsymbol{y}^\top \boldsymbol{y}$): Constant, derivative is $\mathbf{0}$.
- 2nd term ($-2\tilde{\boldsymbol{w}}^\top (\boldsymbol{X}^\top \boldsymbol{y})$): Linear term, derivative is $-2\boldsymbol{X}^\top \boldsymbol{y}$.
- 3rd term ($\tilde{\boldsymbol{w}}^\top (\boldsymbol{X}^\top \boldsymbol{X})\tilde{\boldsymbol{w}}$): Quadratic form with a symmetric matrix ($\boldsymbol{X}^\top \boldsymbol{X}$), derivative is $2(\boldsymbol{X}^\top \boldsymbol{X})\tilde{\boldsymbol{w}}$.

Summing them up gives the gradient:

$$\nabla_{\tilde{\boldsymbol{w}}} L = -2\boldsymbol{X}^\top \boldsymbol{y} + 2\boldsymbol{X}^\top \boldsymbol{X}\tilde{\boldsymbol{w}}$$